Solution to Homework Set 5
Managerial Economics Fall 2011

Conceptual and Computational Questions

1. (10 points) A firm can manufacture a product according to the production function

\[ Q = F(K, L) = K^{3/4}L^{1/4}. \]

(a) Calculate the average product of labor, \( AP_L \), when the level of capital is fixed at 16 units and the firm uses 16 units of labor. How does the average product of labor change when the firm uses 81 units of labor?

When \( K = 16 \) and \( L = 16 \), \( Q = 16 \). Thus, \( AP_L = Q/L = 1 \).

When \( K = 16 \) and \( L = 81 \), \( Q = 16^{3/4}81^{1/4} = 8 \cdot 3 = 24 \). Thus, \( AP_L = 8/27 \).

(b) Find an expression for the marginal product of labor, \( MP_L \), when the amount of capital is fixed at 16 units. Then, illustrate that the marginal product of labor depends on the amount of labor hired by calculating the marginal product of labor of 16 and 81 units of labor.

The marginal product of labor is \( MP_L = 2L^{-3/4} \). When \( L = 16 \), \( MP_L = 2 \cdot 16^{-3/4} = 1/4 \). When \( L = 81 \), \( MP_L = 2 \cdot 81^{-3/4} = 2/27 \).

Thus, as the number of units of labor increases, the marginal product of labor decreases (1/4 > 2/27), holding the level of capital fixed.

(c) Suppose capital is fixed at 16 units. If the firm can sell its output at a price of $100 per unit and can hire labor at $25 per unit, how many units of labor should the firm hire in order to maximize profits?

We must equate the value of marginal product of labor equal to the wage and solve for \( L \). Here, \( VMP_L = 100 \cdot (2L^{-3/4}) = 200L^{-3/4} \). Setting this equal to the wage of $25 gives \( L^{-3/4} = 1/8 \). Solving for \( L \), the optimal quantity of labor is \( L = 16 \).

2. (4 points) Explain the difference between the law of diminishing marginal product and the law of diminishing marginal rate of technical

Let \( G \) be the smaller number between 1.5 + 0.0625*(achieved points) and 6. Round \( G \) exactly to a quarter of a grade to get your grade.
The law of diminishing marginal returns is the decline in marginal productivity experienced when input usage increases, holding all other inputs constant. In contrast, the law of diminishing marginal rate of technical substitution is a property of a production function stating that as less of one input is used, increasing amounts of another input must be employed to produce the same level of output.

5. (3 points) A manager hires labor and rents capital equipment in a very competitive market. Currently the wage rate is $6 per hour and capital is rented at $12 per hour. If the marginal product of labor is 50 units of output per hour and the marginal product of capital is 75 units of output per hour, is the firm using the cost-minimizing combination of labor and capital? If not, should the firm increase or decrease the amount of capital used in its production process?

Since \( \frac{MRTS_{KL}}{w} \neq \frac{MP_K}{r} \), the firm is not using the cost minimizing combination of labor and capital. To minimize costs, the firm should use more labor and less capital since the marginal product per dollar spent is greater for labor:

\[
\frac{MP_L}{w} = \frac{50}{6} > \frac{MP_K}{r} = \frac{75}{12}
\]

7. (8 points) A multiproduct firm’s cost function was recently estimated as

\[
C(Q_1, Q_2) = 75 - 0.25Q_1Q_2 + 0.1Q_1^2 + 0.2Q_2^2
\]

(a) Are there economies of scope in producing 10 units of product 1 and 10 units of product 2?

There are economies of scope, if

\[
C(Q_1, Q_2) < C(Q_1, 0) + C(0, Q_2)
\]

\[
75 - 0.25Q_1Q_2 + 0.1Q_1^2 + 0.2Q_2^2 < 2 \cdot 75 + 0.1Q_1^2 + 0.2Q_2^2
\]

\[
0 < 75 - 0.25Q_1Q_2
\]

With \( Q_1 = Q_2 = 10 \), this holds and therefore there are economies of scope.

(b) Are there cost complementarities in producing products 1 and 2?

Complementarities exist, if the marginal cost of producing one output decreases as the output of the other good is increased. The question therefore is: Does

\[
\frac{\partial C(Q_1, Q_2)}{\partial Q_1}
\]

decrease, if we increase \( Q_2 \)? In our case we have

\[
\frac{\partial C(Q_1, Q_2)}{\partial Q_1} = -0.25Q_2 + 0.2Q_1.
\]
And because 
\[
\frac{\partial}{\partial Q_2} \left( \frac{\partial C(Q_1, Q_2)}{\partial Q_1} \right) = -0.25
\]
is negative, complementarities do exist.

Suppose the division selling product 2 is floundering and another company has made an offer to buy the exclusive rights to produce product 2. How would the sale of the rights to produce product 2 change the firm’s marginal cost of producing product 1? Since \(a = 0.25 < 0\), the marginal cost of producing product 1 will increase if the division that produces product 2 is sold.

9. (5 points) A firm produces output according to a production function \(Q = F(K, L) = \min\{2K, 4L\}\).
   (a) How much output is produced when \(K = 2\) and \(L = 3\)?
   \(Q = \min\{4, 12\} = 4\) units are produced.
   
   (b) If the wage rate is $30 per hour and the rental rate on capital is $10 per hour, what is the cost-minimizing input mix for producing 4 units of output?
   The cost-minimizing mix of \(K\) and \(L\) that produce \(Q = 4\) is \(K = 2, L = 1\).
   
   (c) How does your answer to part b change if the wage rate decreases to $10 per hour but the rental rate on capital remains at $10 per hour?
   Since \(K\) and \(L\) are perfect complements in the production process, the cost-minimizing levels of \(K\) and \(L\) do not depend on the rental rates of \(K\) and \(L\). Therefore, the cost-minimizing levels of \(K\) and \(L\) do not change with changes in the relative rental rates.

10. (5 points) A firm produces output according to the production function \(Q = F(K, L) = 2K + 4L\).
    (a) How much output is produced when \(K = 2\) and \(L = 3\)?
    With \(K = 2\) and \(L = 3\), \(Q = 16\).
    
    (b) If the wage rate is $30 per hour and the rental rate on capital is $10 per hour, what is the cost-minimizing input mix for producing 16 units of output?
    Since the \(MRTS_{KL}\) is 2, that means a company can trade two units of capital for every one unit of labor. This production function does not exhibit diminishing marginal rate of technical substitution. The perfectly substitutability between capital and labor means that only one input will be utilized. Since \(\frac{MP_L}{w} = \frac{4}{30} < \frac{MP_K}{r} = \frac{2}{10}\), the company should hire all capital.
(c) How does your answer to part b change if the wage rate decreases to $10 per hour but the rental rate on capital remains at $10 per hour? The company should hire only labor. 

Problems and Applications

11. (4 points) In an effort to stop the migration of many of the automobile manufacturing facilities from the Detroit area, Detroit’s city council is considering passing a statute that would give investment tax credits to auto manufacturers. Effectively, this would reduce auto manufacturer’s costs of using capital and high-tech equipment in their production processes. On the evening of the vote, local union officials voiced serious objections to this statute. Outline the basis of the argument most likely used by union officials. (Hint: Consider the impact that the statute would have on auto manufacturer’ capital-to-labor ratio.) As a representative for one of the automakers, how would you counter the union officials’ argument? 

An investment tax credit would reduce the relative price of capital to labor. Other things equal, this would increase \( w/r \), thereby making the isocost line more steep. This means that the cost-minimizing input mix will now involve more capital and less labor, as firms substitute toward capital. Labor unions are likely to oppose the investment tax credit since the higher capital-to-labor ratio will translate into lost jobs. You might counter this argument by noting that, while some jobs will be lost due to substituting capital for labor, many workers will retain their jobs. Absent the plan, automakers have an incentive to substitute cheaper foreign labor for U.S. labor. The result of this substitution would be a movement of plants abroad, resulting in the complete loss of U.S. jobs.

Cost Minimizing Architecture

The cost of building a flat roofed house be equal to its surface \( S \) (four walls and a roof). If the customer wants to buy a house with a volume of at least \( V = 1000 \ m^3 \),

\[ \text{CMA1} (4 \text{ points}) \text{ formulate the cost minimization problem and} \]

\[ \begin{align*}
\min_{w, h, l} & S = wl + 2wh + 2lh \\
\text{s.t.} & V = lwh \geq 1000m^3.
\end{align*} \]

CMA1 gives 3 points, if S and V are specified, but \( \min_{(w,h,l)} \) is missing.
CMA2 (10 points) solve for the optimal dimensions length \( l \), width \( w \), and height \( h \).

The Lagrangian reads \( L = 2hw + 2hl + wl - \lambda(hwl - 1000) \). The FOCs with respect to \( l \), \( w \), and \( h \) are

\[
\frac{\partial L}{\partial l} = 0 : \quad 2h + w = \lambda hw \\
\frac{\partial L}{\partial w} = 0 : \quad 2h + l = \lambda hl \\
\frac{\partial L}{\partial h} = 0 : \quad 2w + 2l = \lambda wl.
\]

The first two imply \( w/l = (2h + w)/(2h + l) \) and thus \( w = l \). The second and the third imply

\[
\frac{h}{w} = \frac{2h + l}{2w + 2l} \quad \text{and} \quad \frac{w}{l} = \frac{2h + l}{4l} \quad \Rightarrow \quad h = \frac{l}{2}.
\]

Inserting \( 2h = w = l \) into the constraint \( hwl \geq 1000 \) gives

\[
w = l = 10\sqrt{2} \\
h = 5\sqrt{2}.
\]

Assume that the customer would like his house to be at least twice as long as wide.

CMA3 (10 points) How long, wide, and high is the cost minimizing house with volume 1000 m\(^3\) that fulfills this constraint?

Given that in the above result \( l = w \neq 2w \), the constraint \( l = 2w \) will be binding. Inserting this directly, the Lagrangian becomes \( L = 6hw + 2w^2 - \lambda(h2w^2 - 1000) \). The new FOCs are

\[
\frac{\partial L}{\partial w} = 0 : \quad 6h + 4w = \lambda 4hw \\
\frac{\partial L}{\partial h} = 0 : \quad 6w = \lambda 2w^2.
\]

The two imply

\[
\frac{4hw}{2w^2} = \frac{6h + 4w}{6w} \\
2h = \frac{3h + 2w}{3} \\
\Rightarrow \quad h = \frac{2}{3} w.
\]
Inserting \( l = 2w \) and \( h = 2/3w \) into the volume constraint gives \( 4/3w^3 = 1000 \) and thus:

\[
\begin{align*}
    w &= 10 \sqrt[3]{\frac{3}{4}} = 10.0856 \quad \text{(1 point)} \\
    l &= 20 \sqrt[3]{\frac{3}{4}} = 18.1710 \quad \text{(1 point)} \\
    h &= \frac{20}{3} \sqrt[3]{\frac{3}{4}} = 6.0573 \quad \text{(1 point)}
\end{align*}
\]

**Factor Taxes**

Consider the following convex, decreasing returns to scale technology for producing \( y \):

\[
y = \min\left[K, \sqrt{L}\right].
\]

**FT1** (5 points) What is the profit function \( \pi(y) \), if factors \( K \) and \( L \) cost \( w_K \) and \( w_L \) and are employed optimally? \( y \) is sold at the price \( p \).

If nothing is to be wasted, the production function implies \( y = K = \sqrt{L} \) and thus:

\[
\begin{align*}
    K &= y \quad \text{1 point} \\
    L &= y^2 \quad \text{2 points}
\end{align*}
\]

Profits are:

\[
\pi(y) = py - w_K y - w_L y^2. \quad \text{2 points}
\]

**FT2** (3 points) What is the optimal production plan \((y, K, L)\) if \( p = 3 \) and \( w_K = w_L = 1 \).

In order to maximize \((3 - 1)y - y^2\), we solve the FOC \( 2 = 2y \) and conclude that \( y = 1 \) is optimal. The optimal production plan is:

\[
\begin{align*}
    y &= 1 \quad \text{1-2 right results: 1 point} \\
    K &= 1 \quad 3 \text{ right results: } 2 \text{ points} \\
    L &= 1
\end{align*}
\]

**FT3** (3 points) What is the optimal production plan \((y, K, L)\) if \( w_K \) is increased to 2 (for example by a capital tax).
In order to maximize \((3 - 2)y - y^2\), we solve the FOC \(1 = 2y\) and conclude that \(y = \frac{1}{2}\) is optimal. The optimal production plan is

\[
\begin{align*}
  y &= \frac{1}{2} \\
  K &= \frac{1}{2} \\
  L &= \frac{1}{4},
\end{align*}
\]